Hénon maps

The Hénon maps are a generalization of the function $z \mapsto z^2 + c$ to two complex variables. The Hénon map $H : \mathbb{C}^2 \to \mathbb{C}^2$ is given by

$$H \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x^2 - ay + c \\ x \end{pmatrix}, \quad \text{with inverse} \quad H^{-1} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ \frac{1}{a}\left(y^2 + c - x\right) \end{pmatrix},$$

where a and c are both complex parameters.

Related sets

- K^+ and K^- are the sets of points with bounded orbits under H and H^{-1}, respectively.
- The filled Julia set is $K = K^+ \cap K^-$.
- J^+ and J^- are the boundaries of K^+ and K^-, respectively.
- $J = J^+ \cap J^-$ is the Julia set.

Related functions

Let

$$G^+ \begin{pmatrix} x \\ y \end{pmatrix} = \lim_{n\to\infty} \frac{1}{2^n} \log^+ \left\| H^n \begin{pmatrix} x \\ y \end{pmatrix} \right\|$$
$$G^- \begin{pmatrix} x \\ y \end{pmatrix} = \lim_{n\to\infty} \frac{1}{2^n} \log^+ \left\| H^{-n} \begin{pmatrix} x \\ y \end{pmatrix} \right\|.$$

G^+ and G^- measure how quickly iterates of H (the Hénon map) and H^{-1} go to infinity.

The surfaces we studied were

$$X_t = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{C}^2 \;\middle|\; G^+ \begin{pmatrix} x \\ y \end{pmatrix} = G^- \begin{pmatrix} x \\ y \end{pmatrix} = t \right\},$$

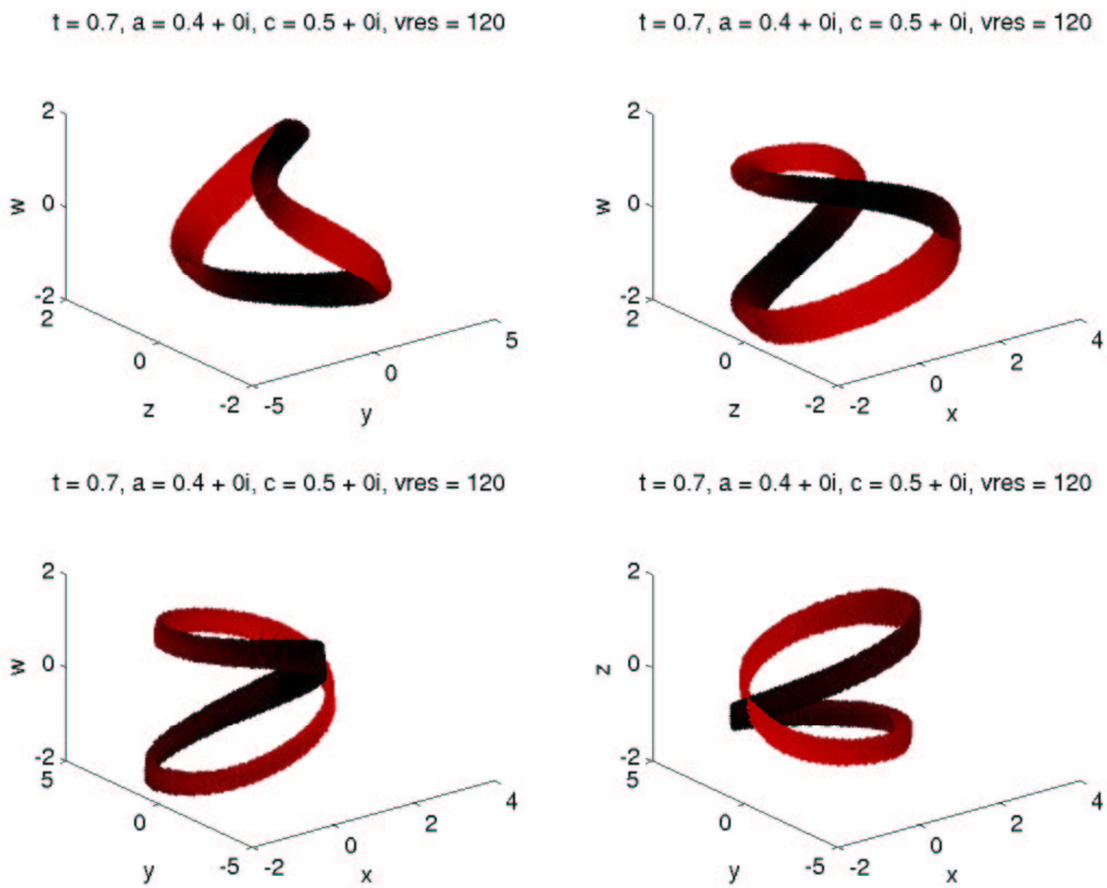
where t is a real nonnegative parameter. As $t \to 0$, X_t approaches the Julia set J.

What is known about X_t

For large t, it is known that X_t is homeomorphic to a torus (and as a result has Euler characteristic 0).

Why: If x and y are both large, then $G^+\binom{x}{y} \approx \log|x|$ and $G^-\binom{x}{y} \approx \log|y|$, so if t is large, then X_t consists of points where $\log|x| \approx \log|y|$, which means that $|x| \approx |y| \approx e^t$, and a point of X_t is determined by the arguments of x and y. This roughly describes a torus.

What does X_t look like?



Strategy for computing X_t

- **Goal:** Write a program, Cubism, which, given particular values for a, c, and t, generates an approximation of X_t and then uses the approximation to compute the Euler characteristic of each connected component of X_t.

- Take a region R in $\mathbb{R}^4$ ($\mathbb{C}^2$), subdivide this region into many four-dimensional cubes, and determine which cubes contain a part of the surface X_t.

- The collection of cubes S that are found is the desired approximation of X_t.

Strategy for analysis of X_t

- In the case of the surfaces X_t, we can classify the connected components of the surfaces by their Euler characteristics.
- Since S should be a thickening of X_t, we can compute the Euler characteristic of X_t by computing the Euler characteristic χ of S using the formula

$$\chi(S) = \sum_{k=0}^{4} (-1)^k n_k,$$

where n_k is the number of k-dimensional faces in S.
- Also, we color the three-dimensional projections of X_t by using the missing dimension.

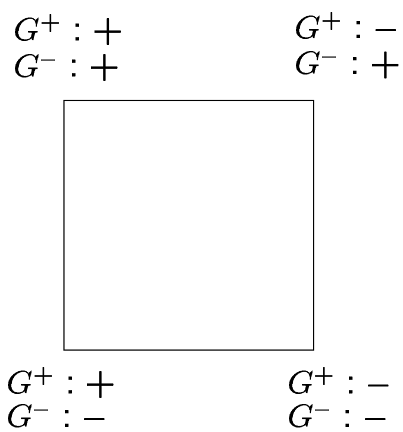
Finding S

- Make some subdivision of R into four-dimensional cubes.
- Run through each two-dimensional face that touches at least one four-dimensional cube in R.
- The intersections of a two-dimensional face with the sets $c^{+} = \{G^{+} = t\}$ and $c^{-} = \{G^{-} = t\}$ are each one-dimensional curves. If these curves intersect each other in the face in question, then all 4 four-dimensional cubes which touch this face are added to S.
- How to *efficiently* determine whether or not c^{+} intersects c^{-} in a particular face?

Vertex decisions

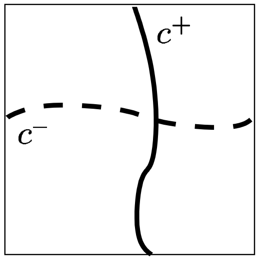
- Examine the signs of the functions $G^+ - t$ and $G^- - t$ at the vertices of the face.
- Certain combinations of signs at the vertices can ensure that c^+ will intersect c^- on a face.

For example, say that the signs on a face are as below:



Vertex decisions: a "yes" face

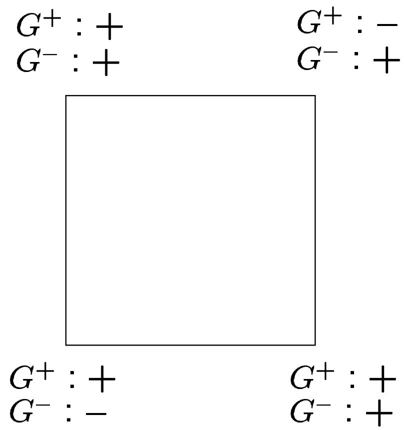
In this case, we can be sure that c^+ and c^- intersect in that face, looking something like this:



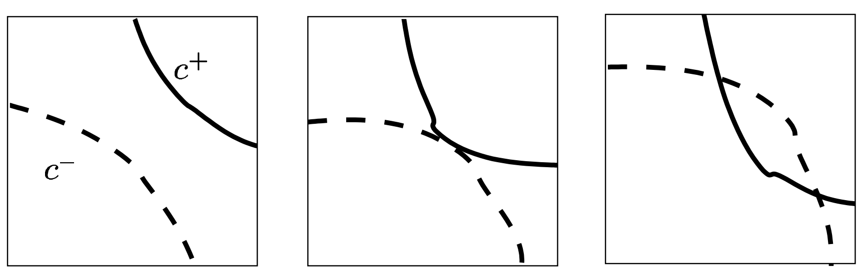
In the case above and in similar cases (when we can be sure that there is an intersection), we mark the face in question as a "yes".

Vertex decisions: a “maybe” face

On the other hand, this combination of signs



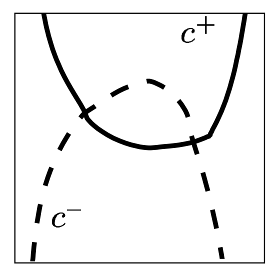
could manifest itself in any of the following situations:



Problems with vertex decisions

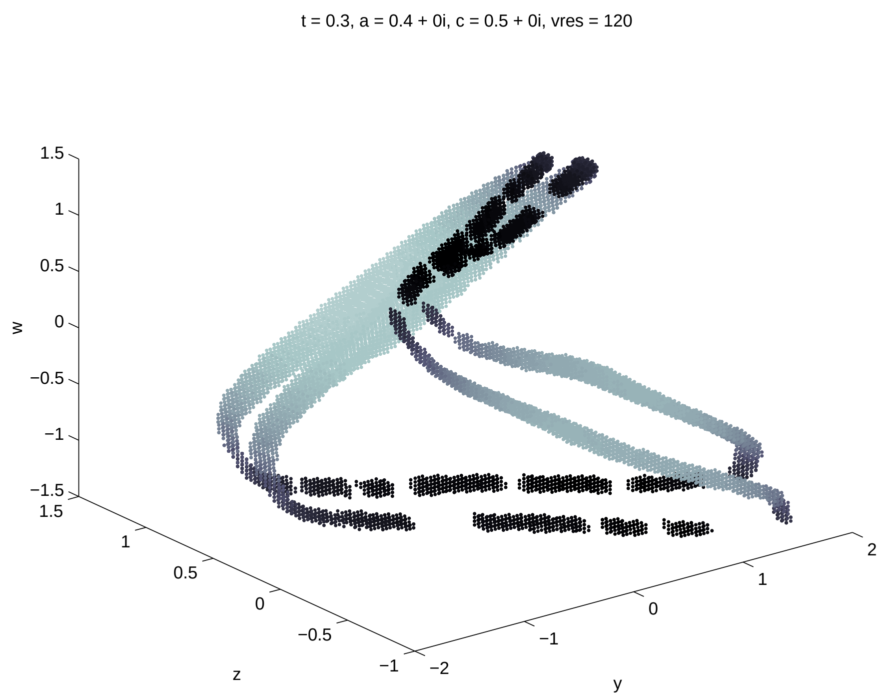
If this algorithm marks a face as a "yes", then c^{+} and c^{-} definitely intersect within the face. But even if we consider S to be the set of all "yes" cubes *and* all "maybe" cubes, we still miss cubes that should be in S.

Either $G^{+} - t$ or $G^{-} - t$ could change signs twice, resulting in signs at the vertices that are identical to the case where the function does not change sign at all. (See the diagram at right.)



A picture with gaps

Using only the vertex decision method resulted in approximations of X_t that had gaps in them (cubes that should have been in S but weren't):



Getting around problems with vertex decisions

- Given an infinite amount of time (and/or computational power), we could fix the problem by making a finer subdivision of R.
- But Cubism takes time $O(N^4)$, if N is the number of subdivisions on a side, so it wasn't reasonable to simply increase N.
- Instead, we needed a better algorithm.

Newton's Method

- Using a two-dimensional version of Newton's method, we attempt to find a common root of the functions $G^{+} - t$ and $G^{-} - t$, with the initial point in the center of the face.

- We iterate Newton's method a fixed number of times (5) and then try to determine whether Newton's method is converging to a root.

- If it is, then there is an intersection of c^{+} and c^{-} in the face, and we mark the face as a "yes".

Determining whether Newton's Method is converging

- It is difficult to determine whether or not Newton's method is really converging to a root within the face. (Could mistake convergence of Newton's Method to a root on the face in question for converging to a root that is on another face.)

- Our criteria for convergence: if the final iterate of Newton's method is still within the face it started in, *and* both G^+ and G^- are within ϵ of t, then we mark this face as a "yes". (Set $\epsilon = .01$.)

- The second condition attempts to ensure that Newton's method is in fact converging, and the first condition attempts to ensure that Newton's method is converging to a root inside the face it started in.

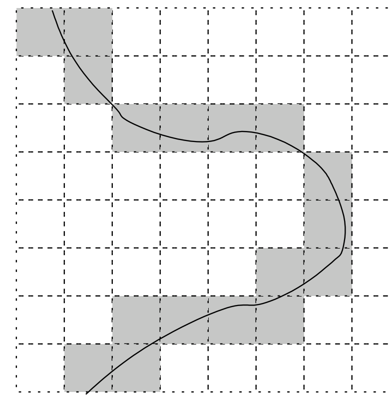
Subdivision and Refinement

- If we choose an initial subdivision N (so that each side of R is divided into N subdivisions), then we end up with N^4 cubes.

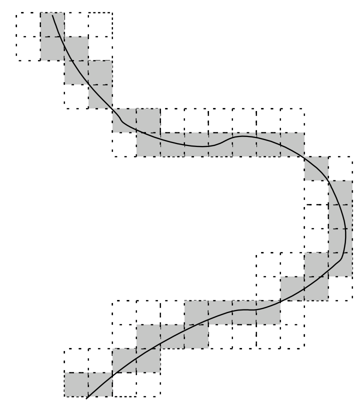
- A larger N results in a better approximation of the surface X_t, so we want to run Cubism for as large an N as possible.

- We use a method that allows us to start with a reasonable initial subdivision (i.e., not too large), and then "refine" the resulting cubes into smaller and smaller cubes.

Subdivision and Refinement, cont.



Start with some initial subdivision, and let S_0 be the resulting set S of cubes marked as a "yes" or a "maybe". In the lower-dimensional diagram at right, S_0 is the set of shaded boxes:



For each four-dimensional cube in S_0, perform another subdivision, and let S_1 be the resulting collection of cubes. Repeat as desired. At right, S_1 is the set of shaded cubes:

Subdivision and Refinement, cont.

- Note that if subdivisions $N_0, N_1, \ldots, N_k$ are used, then the final thickening of X_t is the set S_k.
- In theory, we should obtain the same thickening of X_t by doing this procedure or by choosing the sole subdivision to be $N_0 \cdot N_1 \cdot N_2 \cdots N_k$.
- In practice, this doesn't quite work—if both the vertex decision algorithm and the Newton's method algorithm mark a face a "no" that should be a "yes" or "maybe", then not only will that cube not be included in S, but that cube will also not be refined (further subdivided).