REU 2001 Report: Geometry of Numbers.

Konstantin Rybnikov *

August 24, 2001

Abstract

During the summer of 2001 undergraduate students Michael Greene (Harvard), Jody Radowicz (North Central), Sarah Crown (Bryn Mawr) and Kirsten Wikelgren (Harvard) studied geometry of homogeneous and inhomogeneous quadratic forms over integers. I was helped in this project by Cornell Ph.D. students Todd Kemp and Franco Saliola. Students worked in the following directions: geometry and combinatorics of perfect Delaunay polytopes, connections between extreme L-types and perfect forms, and the symmetries of the Voronoi polyhedron $\Pi(n)$.

Keywords: point lattice, Delaunay polytope, L-type, perfect form, quadratic forms over integers, Voronoi polyhedron

1 Introduction

Positive quadratic forms (referred to as PQFs) in n indeterminates form a closed cone $\mathfrak{P}(n)$ of dimension $N = \frac{n(n+1)}{2}$ in $\mathbb{R}^N$. The interior of $\mathfrak{P}(n)$ consists of positive definite quadratic forms (PDQFs) of rank n. PDQFs serve as analytic representations of *point lattices*. Let $\mathbf{v}_1, \ldots, \mathbf{v}_1$ be a coordinate frame in Euclidean space $\mathbb{E}^n$. A *lattice* of points with basis $[\mathbf{v}_1, \ldots, \mathbf{v}_n]$ is the set of all integral linear combinations of $\mathbf{v}_1, \ldots, \mathbf{v}_n$. If $G = (g_{ij})$ is the Gramm matrix of a PDQF, then the basis $[\mathbf{v}_1, \ldots, \mathbf{v}_n]$ of the corresponding lattice is defined (up to isometry) by $g_{ij} = \mathbf{v}_i \cdot \mathbf{v}_j$. Conversely, given a basis $[\mathbf{v}_1, \ldots, \mathbf{v}_n]$, the Gramm matrix of the corresponding PDQF is $[\mathbf{v}_1, \ldots, \mathbf{v}_n]^T[\mathbf{v}_1, \ldots, \mathbf{v}_n]$. Thus, there is a one-to-one correspondence between congruence classes of n-dimensional lattices and integral equivalence classes (i.e. with respect to $GL(n, \mathbb{Z})$-conjugation) of positive quadratic forms. For basic results of the theory of lattices and PDQFs and their applications see Ryshkov and Baranovskii (1978), Gruber and Lekkerkerker (1987), Erdös, Gruber and Hammer (1989), Conway and Sloane (1999).

$GL(n, \mathbb{Z})$ acts pointwise on the space of quadratic forms $\mathbb{R}^N$. The *L-type partition and perfect partition* are important polyhedral subdivisions of $\mathfrak{P}(n)$, invariant with respect to the action of $GL(n, \mathbb{Z})$ on $\mathbb{R}^N$. These partitions have been intensively studied in geometry of numbers and combinatorics.

*Department of Mathematics, Cornell University, USA. E-mail: rybnikov@math.cornell.edu

Definition 1.1 *Let S be a discrete set of points in $\mathbb{E}^n$. A convex polytope P in $\mathbb{E}^n$ is called a Delaunay cell of the system of points S if:*

1) all vertices of P belong to S,

2) there is a sphere circumscribed about P (the Delaunay sphere *of P),*

3) no points of S, except for the vertices of P, lie inside or on its Delaunay sphere.

If S is not pathological, Delaunay cells form a convex face-to-face tiling of $conv\, S$ that is defined uniquely by S. Delaunay tilings have many applications in computational geometry, mesh generation, the theory of lattices, mathematical crystallography, etc.

Definition 1.2 *Lattices L_1 and L_2 belong to the same L-type if their Delaunay tilings are affinely equivalent (the notion of L-type is, in fact, due to Voronoi (1908-09)).*

The L-type of a PDQF is defined as the L-type of its lattice. L-type domains are open pointed polyhedral cones in $\mathbb{R}^N$ ($N = \frac{n(n+1)}{2}$). In each dimension there are only finitely many L-type domains (Voronoi, 1908-09). Each L-type domain is, of course, the union of infinitely many disjoint open convex cones that are equivalent with respect to $GL(n, \mathbb{Z})$ acting pointwise on $\mathbb{R}^N$. L-type domains tile $\mathfrak{P}(n)$.

In the above definition instead of varying the lattice one could fix the lattice and vary the metric form with the only requirement that it remains positive semidefinite. Then two PDQFs belong to the same *L-cone* if there is a smooth path joining them in the space of parameters such that all forms on this path have the same Delaunay tiling of $\mathbb{Z}^n$. Two PDQF belong to the same *L-type* if their Delaunay tilings are affinely equivalent. *In this report we will always asume that our lattice is $\mathbb{Z}^n$ and we will vary the quadratic forms.*

The notions of Delauny tiling and L-type are extremely important in the study of extremal and group-theoretic properties of lattices. For example, the analysis of the Delaunay tiling of the famous Leech lattice conducted by Conway, Sloane and Borcherds showed that the 23 "deep holes" in the lattice (Delaunay cells of radius equal to the covering radius of the lattice) are in one-to-one correspondence with 23 even unimodular 24-dimensional lattices (Niemeier's list minus Leech lattice) that, in turn, give rise to 23 "gluing" constructions of the Leech lattice from root lattices. Barnes and Dickson (1967, 1968) and, later, Delaunay et al. (1969, 1970) proved that the closure of each N-dimensional L-type domain has at most one local minimum of the covering density, and if such a minimum exists and lies in the interior of the domain, the group of $GL(n, \mathbb{Z})-$automorphisms of the domain maps this form to itself. Using this approach Delaunay, Ryshkov and Baranovskii (1963, 1976) found the best lattice coverings in $\mathbb{E}^4$ and $\mathbb{E}^5$. The theory of L-types also has numerous connections to combinatorics and, in particular, to cuts, hypermetrics, and regular graphs (see Deza et al. (1997)).

The notion of L-type partition of the cone of PDQFs is closely related to the theory of *perfect forms* originated by Korkine and Zolotareff (1873), although the nature of this relationship is not well understood. Let $f(\mathbf{x},\mathbf{x})$ be a PDQF. The *arithmetic minimum* of $f(\mathbf{x},\mathbf{x})$ is the minimum of $f(\mathbf{x},\mathbf{x})$ on $\mathbb{Z}^n$. The integral vectors on which this minimum is attained are called the *representations of the minimum*, or the minimal vectors of $f(\mathbf{x},\mathbf{x})$: these vectors have the minimal length among all vectors of $\mathbb{Z}^n$ when $f(\mathbf{x},\mathbf{x})$ is used as the metrical form.

Definition 1.3 *$f(\boldsymbol{x},\boldsymbol{x})$ is called* perfect *if it can be reconstructed from all representations of its arithmetic minimum.*

In other words, a PDQF with the arithmetic minimum m and the set of minimal vectors $\{\mathbf{v}_k|\, k = 1, ..., 2s\}$ is perfect if the system

$$\sum_{i,j=1}^{n} a_{ij} v_{ik} v_{jk} = m,$$

where $k = 1, ..., 2s$, has a unique solution (a_{ij}) in the space of symmetric matrices $\mathbb{R}^N$ (of course, there must be at least $\frac{n(n+1)}{2}$ non-collinear minimal vectors). In each dimension there are only finitely many perfect forms up to $GL(n, \mathbb{Z})$-equivalence (Voronoi, 1908-09). Intuitively, perfect lattices are those that have a large supply of minimal vectors, although a perfect lattice in dimension higher than 8 is not always spanned by its minimal vectors (Ryshkov, 1973). A perfect form $f(\mathbf{x},\mathbf{x})$ can be described as a hyperplane in $\mathbb{R}^N$ that contain $N + 1$ affinely independent integer points whose coordinates are the images under the Veronese-Voronoi mapping $\mathcal{V} : (x_1, \ldots, x_n) \rightarrow (x_1^2, \ldots, x_n^2, x_1x_2, \ldots, x_{n-1}x_n)$ of the representations of the minimum of $f(\mathbf{x},\mathbf{x})$. Perfect forms play an important role in lattice sphere packings. A theorem of Voronoi (1908) says that a form is *extreme*—i.e., a maximum of the packing density—if and only if it is perfect and *eutactic* (see Coxeter (1951) or Conway, Sloane (1988) for the proof). The notion of *eutactic form* arises in the study of the dense lattice sphere packings and is directly related to the notion of perfect form. The dual (also called reciprocal) of $f(\mathbf{x},\mathbf{x})$ is a form whose Gramm matrix is the inverse of the Gramm matrix of $f(\mathbf{x},\mathbf{x})$. The dual form is normally denoted by $f^*(\mathbf{x}, \mathbf{x})$. A form $f(\mathbf{x}, \mathbf{x})$ is called *eutactic* if the dual form $f^*(\mathbf{x}, \mathbf{x})$ can be written as $\sum_{k=1}^{s} \alpha_k (\mathbf{v}_k \bullet \mathbf{x})^2$, where $\{\mathbf{v}_k|\, k = 1, ..., s\}$ is the set of mutually non-collinear minimal vectors of $f(\mathbf{x}, \mathbf{x})$, and $\alpha_k > 0$.

Points $\mathcal{V}(\mathbb{Z}^n)$ in $\mathbb{Z}^N$ are called Voronoi points and their convex hull in $\mathbb{R}^N$ is called the *Voronoi Polyhedron* $\Pi(n)$. The supporting hyperplanes for the facets of this polyhedron are perfect forms, in the sense that they are defined by equations $trace(FP) = mar$, where F is the Gramm matrix of a perfec form and P is the Gramm matrix of a Voronoi point (the variable).

A cone in $\mathbb{R}^N$ spanned by the images of the minimal vectors of a perfect form is called a perfect cone. The union of all (closed) perfect cones corresponding to forms integrally equivalent to f is called the perfect domain of f. For each perfect cone there are infinitely many $GL(n, \mathbb{Z})$-equivalent ones, so the perfect domain of f consists of infinitely many equivalent perfect cones, just like an L-type domain consists of infinitely many convex cones. A fundamental theorem of Voronoi (1908) in the interpretation of Delaunay and Ryshkov (1968) says that the cone of PSQFs is tiled face-to-face by perfect cones. A facet of this tiling is called a perfect wall. Voronoi proved that there are only finitely many non-equivalent perfect forms. Therefore, there is a finite set of perfect cones in $\mathbb{R}^N$ such that each form in n variables is equivalent to a form lying in one of these domains. Voronoi gave an algorithm finding all perfect domains for given n. This algorithm is known as Voronoi's reduction with perfect forms. For the computational analysis of his algorithm and its improvements see Martinet (1996). Voronoi's algorithm was successfully applied by him and other researchers in searching for perfect forms in lower dimensions, i.e. $n \leq 9$ (see Voronoi (1908-09), Barnes (1957), Martinet (1996)). Perfect forms have been completely classified in dimensions $n \leq 7$, but already for $n = 9$ there are millions of them (see Martinet (1996)).

The notion of perfect Delaunay cell is an inhomogeneous analog of the notion of perfect form.

Definition 1.4 *Let $P \subset \mathbb{R}^n$ be a Delaunay polytope in $\mathbb{Z}^n$ for PQF* f. *P is called perfect if the only ellipsoid circumscribed about P is the one defined by the form* f. *In this case the inhomogeneous quadratic form defining this ellispoid is called* perfect ellispoid.

The definition of perfect ellipsoid clearly reminds us of the definition of perfect form; in fact, perfect ellipsoid is an exact inhomogeneous analog of the notion of perfect form. The minimal possible number of vertices of a perfect Delaunay cell is $\frac{n(n+1)}{2} + n$. Perfect Delaunay polytopes (Deza et al. (1997) call them extreme) are important not only to the theory of lattices, but also to the theory of hypermetrics started by analists such as Scheunberg (1937): perfect Delaunay cells are in one-to-one correspondence with extreme rays of the cone of hypermetrics (see Deza et al. (1997)). In our research we are primarily interested in geometric aspects of perfect Delaunay cells and ellipsoids.

An L-cone of dimension 1 is called extreme. Extreme L-types that are interior to the cone of PQFs are very rare. They first occur in dimension 4. kD_4, $k > 0$ is an extreme L-type. Obviously, if the Delaunay tiling of f contains a perfect Delaunay cell it is unstable under any perturbation of f different from scaling and, therefore, lies on an *internal* extreme ray of the L-type partition of the cone of PQFs. Thus, any new perfect Delaunay cell would provide for a new example of an extreme L-type. *Presently, only finitely many examples of extreme L-types and perfect ellipsoids are known.*

The significance of extreme L-types is much due to their relation to the structure of Delaunay and Dirichlet-Voronoi tilings. Dirichelt-Voronoi polytopes of higher-dimensional lattices are important to the theory of quasicrystalls, coding theory, information quantization, etc (see Conway, Sloane (1999), Senechal (1995)). The significance of extreme L-types for geometry of lattices is illustared by the following propositions.

Theorem 1.1 *(Ryshkov (1998), Erdahl, 2000b) The Voronoi polytope for any PQF f is the Minkowski sum of Voronoi polytopes for quadratic forms that lie on the extreme rays of the L-cone of f.*

Theorem 1.2 *(Erdahl, 2000b) The Delaunay tiling for any PQF is the intersection of the Delaunay tilings of forms lying on the extreme rays of the L-cone of f.*

These propositions are dual formualtions of the same statement. Therefore, Voronoi polytopes for extreme L-types are "building blocks" for Voronoi polytopes of arbitrary forms, while Delaunay tilings for extreme rays of the L-cone of f are coarsenings of the Delaunay tiling of f.

Very few perfect Delaunay cells are known. Good examples of such cells are Gosset polytopes in dimensions 6 and 7 (see Coxeter (1995), Erdahl (1987), Deza et al. (1995)), and it is known that there are no perfect Delaunay polytopes in dimensions less than 5. Other examples include two 16-dimensional polytopes in Barnes-Wall lattice BW and its sublattice, three 15-dimensional polytopes in sections of BW, two polytopes in 22 and 23 dimensional sections of the Leech lattice (Deza et al. (1995), Deza et al. (1992)). Some of these examples are manifestations of such phenomena as extreme sets of equiangular lines and extreme spherical two-distance sets that have been intensively studied in algebraic combinatorics (see Lemmens, Seidel (1973)).

Erdahl and Rybnikov (very recently) found an infinite series of perfect Delaunay cells Υ_n not coming from equiangular lines or two-distance sets. Erdahl's series starts from Gosset polytope in E_6. The construction of this series is based on the infinite series of Delaunay simplexes of relative volume $n-3$ found by Erdahl and Rybnikov (2000a). This is the best known infinite series of big Delaunay simplexes; it improves upon Ryshkov's (1973) series of Delaunay simplexes of volume r in dimension $n = 2r + 1$ (it is interesting that Ryshkov's series of big simplexes is also related to his series of perfect lattices not generated by its perfect vectors, 1973). Polytope Υ_n are constructed by supplementing the vertices of the simplex of volume $n - 3$ in some very special way this construction generalizes the embedding of the simplex of volume $n - 3$ in to the Gosset polytope in $\mathbb{R}^6$.

Delaunay foresaw the importance of Delaunay simplexes of non-fundamental volume: in his seminal work of 1937 he asked about possible volumes of Delaunay simplexes. Delaunay simplexes are very special cases of empty lattice simplexes that have been attracting interest of mathematicians due to their importance in integer programming (e.g. see Ziegler, Haase (2000)). While there are pretty sharp results on empty lattice simplexes, not much is known about Delaunay emptry lattice simplexes. The volume of an empty lattice simplex can be arbitrary for $n \geq 4$ (Ziegler et al.(2000)), but the volume of a Delaunay n-simplex is, obviously, bounded from above by $\frac{n!}{2}$. It is not even known if the maximal volume of a Delaunay simplex grows linearly, polynomially, or exponentially in the dimension. The biggest Delaunay simplex we know of lives in the Leech lattice and has volume $85n$ ($n = 24$), but the best infinite series known is linear: $n - 3$.

1.1 Problems Posed

In this REU project students were offered problems about perfect Delaunay polytopes and perfect forms. The first problem was to prove an appealing conjecture stating that a certain, infinite in n, series of $\mathbb{Z}^n$-polytopes is indeed, a series of perfect Delaunay polytopes. The second probelm was to verify the conjecture of Ryshkov stating that the group of the Voronoi Polyhedron is the homomorohic image of $GL(n, \mathbb{Z})$.

Robert Erdahl, assisted by Konstantin Rybnikov, constructed the following infintie series of convex polytopes Υ_n for all n. We call them *supertopes*. Below are the vertices of Υ_n in $\mathbb{Z}^n$.

$$[0^n]$$
$$[-1, 0^{n-1}; 1]$$
$$[1^{n-1}; n-3]$$
$$[0, 1^{n-2}; n-4]$$
$$[1^2, 0^{n-3}; 1]$$

$$[1, 0^{n-1}; 0]$$

Erdahl and Rybnikov realized that the matrix of the quadratic part of the ellispoid circumscribed about Υ_n should have form

$$Q = \begin{pmatrix} d & m & m & m & \dots & e \\ m & d & m & m & \dots & e \\ m & m & d & m & \dots & e \\ m & m & m & d & \dots & e \\ \dots & \dots & \dots & \dots & \dots & \dots \\ e & e & e & e & \dots & b \end{pmatrix}$$

and that the linear part must have form $(d, \dots, d; 1)$, where d is the diagonal coefficient of the matrix of the quadratic part if $R = 1$. For $n = 6, 7, 8$ Erdahl and Rybnikov found the matrices and they had the conjectured form.

1.2 Establishing formulae for the ellipsoids circumscribed about Υ_n. Perfection and Delaunay property

Michael Greene found that for $n = 9 - 28$ there is an ellipsoid circumscribed about $Upsilon_n$ and it has the conjectured form. Using this information Todd Kemp and Jody Radowicz proved that for all $n > 5$ the following ellipsoids are circumscribed about polytopes Υ_n.

Theorem 1.3 *Equation* $\mathbf{x^tQx+Lx = 1}$ *defines an ellipsoid circumscribed about polytope Υ_n. Here Q is a symmetric positive matrix of the above form where $d = q_{ii} = \frac{2-5n+n^2}{-6+3n+n^2}$ for $i < n$, $b = q_{nn} = \frac{4-5n+n^2}{-6+3n+n^2}$, $m = q_{ij} = \frac{-2-3n+n^2}{-6+3n+n^2}$ for $i < j < n$, $e = q_{in} = \frac{12-2n+n^2}{-6+3n+n^2}$ for $i \neq n$; L is a linear functional with $L_i = d$ and $L_n = 1$.*

Franco Saliola proved that for each n the above ellipsoid is the only ellipsoid circumscribed about Υ_n.

Michael Greene and Kirsten Wickelgren were trying to prove that polytopes Υ_n are Delaunay. *Michael proved this for $n = 7$ by extensive computer check.* His method can only be used for dimensions less than 9 due to high computational complexity. Mike and Kirsten reduced the problem to a simplier-looking diophantian inequalities that have not yet been proved.

Michael Greene and Kirsten Wickelgren continue their work on the problem of proving the Delaunay property of Υ_n under my supervision. We if are able to prove the Delaunay property for all n, this result can certainly be published in a top level journal.

1.3 Perfect Delaunay cells and Perfect Forms

The infinite series of polytopes Υ_n generalizes the Gosset polytope G_6 (2_{21} in Coxeter's notation). G_6 is a Delaunay cell for form E_6 corresponding to root lattice E_6. From the point of view of the theory of perfect forms form E_6 belongs to the domain of form E_6^*

which is dual to E_6. In order to understand the geometry of the Delaunay tiling of E_6 it is important to understand the relashionship between forms E_6 and E_6^*.

Let us refer to the quadratic part of the form of the ellipoid circumscribed about Υ_n as to υ_n. There is an infinite series of perfect forms that, in some sense, generalizes the form E_6^*. This series of form was recently discovered by Erdahl and Rybnikov (2002) and is called TF_n; $TF_6 \equiv E_6^*$. (There is a strong suspicion that this series is equivalent to the well-known series h_n of M. Anzin (1991), but neither Rybnikov not Anzin were able to prove this so far.) *It was conjectured by Erdahl that the form TF_n is to υ_n, as E_6^* to E_6. Kirsten Wickelgren and Jody Radowitz have been working in this direction.* If Erdahl's conjecture is true, υ_n must be an interior point of the cone of TF_n. *Jody Radowitz for this for odd n and numerically verified this conjecture for even $n < 28$.* For odd n the coefficients in front of the forms corresponding to perfect vectors of TF_n that are common with D_n are $\frac{n-5}{n-1}$ and the coefficients for all other rank-one forms are $\frac{4}{n-1}$. Jody is now working on the proof for the even case. It follows from Jody's computations that as n goes to infinity, υ_n approaches the wall between the perfect domains of TF_n and D_n. Understanading the relashionship between the representations of the arithmetic minimum of TF_n and υ_n is crucial for proving that Υ_n is a Delaunay polytope. *Jody's work may lead to a better version for the Gramm matrix for the quadratic form of the ellipsoid.* The present matrix given in Theorem 1.3 is far from optimal for practical computations.

1.4 Combinatorics of Υ_n. Delaunay property of Υ_n

Sarah Crown worked on the geometric and combinatorial properties of supertopes. This work was motivated by the following considerations. First, knowledge of combinatorics of Υ_n would help us to prove the Delaunay property

Second, it would be very nice to have a geometric construction for supertopes that is more meaningful than just the list of vertices.

If Υ_n is Delaunay, its vectors must satisfy certain conditions. Sarah checked this condition for $n = 7 - 9$ and did not find any contradiction with Delaunay property.

The following theorem is a very important tool for working with the Delaunay tilings of $\mathbb{Z}^n$.

Two integral vecotors are said to be in the same parity class if they are equivalent *mod* 2.

Theorem 1.4 *(Voronoi, Baranovskii) Let vectors $\pm\mathbf{v}_1, \dots, \pm\mathbf{v}_k \in \mathbb{Z}^n$ belong to the same parity class. Suppose also they are the only minimal vectors w.r.t form f in their parity class. Then the Delauny tiling for f has a cell of dimension at most k which has $2k$ vertices and whose diagonals intersect at a common midpoint. Conversly, if all the segments joining the vertices of a face F of the Delaunay tiling for f are interior to F, then (1) F has $2k$ vertices, (2) all these segments belong to the same parity class and are the only minimal vectors in this class, (3) all these segments intersect at a common midpoint.*

Sarah Crown and Mike Green showed that for $n = 7 - 15$ all parity classes are represented as edges or diagonals of Υ_n. Sarah also established that for dimensions

$n = 7 - 9$ the only possible faces are simplices and cross-polytopes. She conjectured that the number of cross-polytopal facets is always twice the dimension.

Sarah tried to find an "inductive approach" to supertopes, namely a geometric method that takes Υ_{n-k} and produces Υ_n. As an example one can mention Delaunay-Barnes lamination method and its generalizations (see Conway, Sloane (1988); Baranovski (1991)). She figured out that there is no way to build Υ_n from Υ_{n-1} so that Υ_{n-1} can be regarded as a projection of Υ_n along some 1-subspace.

Strongly regualr graphs are very imposrtant objects of algenraic combinatorics. Distance-regular graph (also called a metric association scheme) is a far-reaching generalization of the notion of strongly regular graph. The 1-skeleton of Gosset polytope Υ_6 (2_{21} in Coxeter's notation) is a well-known "sporadic" strongly regular graph called Schlafli graph. Rybnikov conjectured that the 1-skeletons of Υ_n are strongly regualr graphs. Sarah proved that for $n = 7, 8, 9$ the 1-skeleton of Υ_n is NOT a distance-regular graph.

If we are able to make progress in our research on combinatoircs of supertopes, a paper by Erdahl, Rybnikov and Crown on this topic is likely to apear. The title could be "Supertopes as a Generalization of Gosset Polytope 2_{21}"

1.5 Perfect Forms and Voronoi Polyhedron

Voronoi Polyhedron $\Pi(n)$ is the convex hull of $\mathcal{V}(\mathbf{Z}^n)$, where V is the Veronese-Voronoi mapping from $\mathbf{Z}^n$ to $\mathbb{R}^N$, $N = \frac{n(n+1)}{2}$. Voronoi Polyhedron is not Dirichlet-Voronoi polytope which is the set of points that are at least as close to the origin as to any other lattice point. (Unfortunately the term Voronoi polyheron is often used to refer to Voronoi polytope.) The facets of Voronoi polyheron can be though of as perfect forms. As was explained in the introduction perfect forms are central to the problem of the densest *lattice packings* with balls and to reduction methods for low n.

It is easy to prove that conjugation by an element from $GL(n, \mathbb{Z})$ is a an element of $GL(N, \mathbb{Z})$. Thus Veronese-Voronoi mapping V defines a homomorphism from $GL(n, \mathbb{Z})$ to $GL(N, \mathbb{Z})$. Let us denote this homomorphism also by V and the resulting subgroup by $\mathcal{V}(GL(n, \mathbb{Z}))$. $\mathcal{V}(GL(n, \mathbb{Z}))$ preserves $\Pi(n)$, as it sends Voronoi points to Voronoi points. The following natural question arises. Are there any linear transformations of $\Pi(n)$ that do not belong to $\mathcal{V}(GL(n, \mathbb{Z}))$? This problem was posed by S. Ryshkov at the end of 1970s. He announced a proof of this conjecture, but later withdrew his claim. Kirsten Wickelgren worked on this question. With some help from Rybnikov she found a relatively elementary proof of Ryshkov's conjecture. She is currently writing a paper on the result. The paper will be submitted to a mathematical research journal. In the process of our work on the conjeture a few new ideas were investigated. Although these ideas are not directly related to the proof, they are interesting on their own and should be further explored. As an example I would mention an observation that for $n \leq 6$ perfect forms can be distinguished by the volumes of corresponding parallelograms in the space of parameters.

2 Summary

The Geometry of Numbers group worked on perfect quadratic n-forms and perfect Delaunay polytopes in $\mathbb{Z}^n$. The group *derived formulae* for the ellipsoids circumsribing polytopes that were constructed by Erdahl and Rybnikov and *proved that they are perfect* for all $n \geq 6$. A formula expressing the quadratic part of this ellipsoid as a positive linear combination of rank one forms corresponding to the representations of the minimum of form TF_n was derived for odd n and conjectured for even n. The proof of this formula for the even case will be completed shortly. The Delaunay property of $\Upsilon_6 = 2_{21}$ was well known (Coxeter). By direct computer check the Delaunay property was proved for $n = 7$. A necessary condition (the parity classes criterion) for the Delaunay property have been checked for dimensions $8 - 15$; all these conditions have been satisfied. A lot of combinatorial information about the supertopes has been gathered. This date requires careful analysis. In the 197os well-known specialists in geometry of numbers M. Shtogrin and S. Ryshkov conjectured that the group of Voronoi polyhedron $\Pi(n)$ is the homomorphic image of $GL(n, \mathbb{Z})$ under Voronoi-Veronese lifting. In particular this conjecture implies that the homomorphic image of $GL(n, \mathbb{Z})$ is the whole linear transformation group of the integral points in the cone of positive semidefinite matrices, i.e. $\mathbb{Z}^N \cap \mathfrak{P}(n)$. Kirsten Wickelgren with help from Rybnikov finally proved this conjecture. Wickelgren is currently writing a paper on this result. The tentative title of the paper is "The Group of Linear Transformations Preserving the Voronoi Polyherdon".

The following bibliography might be a little bit excessive, but is quite comprehensive.

References

[1] M.M. Anzin, On variations of positive quadratic forms (With applications to the study of perfect forms), *Trudy Mat. Inst. Steklov. Discrete geometry and topology.* Vol. 196 (1991), 11–26; translation in *Proc. Steklov Inst. Math.*

[2] E. S. Barnes, The complete enumeration of extreme senary forms. *Philos. Trans. Roy. Soc. London.* Ser. A. **249** (1957), 461–506.

[3] E. S. Barnes, T. J. Dickson, Extreme coverings of n-space by spheres. *J. Austral. Math. Soc.* **7** (1967), 115–127.

[4] Conway, J. H.; Sloane, N. J. A. Low-dimensional lattices. III. Perfect forms. *Proc. Roy. Soc. London Ser. A* **418** (1988), no. 1854, 43–80.

[5] Conway, J. H.; Sloane, N. J. A. *Sphere packings, lattices and groups.* Third edition. Grundlehren der Mathematischen Wissenschaften, **290**. Springer-Verlag, New York, (1999).

[6] H. S. M. Coxeter, *Regular Polytopes*, Dover, NY, 3rd edition (1973).

[7] Coxeter, H. S. M. Extreme forms. *Canadian J. Math.* **3**, (1951). 391–441.

[8] Coxeter, H. S. M. *Kaleidoscopes. Selected writings of H. S. M. Coxeter. Edited by F. Arthur Sherk et al.*. Canadian Mathematical Society Series of Monographs and Advanced Texts. A Wiley-Interscience Publication. John Wiley & Sons, Inc., New York, 1995.

[9] Delaunay [Delone] B. N. Sur la sphère vide, in: *Proceedings of the International Congress of Mathematicians, Toronto, 1924*, University of Toronto Press, Toronto, (1928), 695-700.

[10] Delone [Delaunay] B. N. The geometry of positive quadratic forms, *Uspekhi Mat. Nauk*, **3** (1937), 16-62; **4** (1938), 102-164.

[11] Delone, B. N.; Ryškov, S. S. Solution of the problem on the least dense lattice covering of a 4-dimensional space by equal spheres. (Russian) *Dokl. Akad. Nauk SSSR* 152, (1963), 523–524.

[12] Delone, B. N.; Dolbilin, N. P.; Ryškov, S. S.; Štogrin, M. I. A new construction of the theory of lattice coverings of an n-dimensional space by congruent balls. (Russian) *Izv. Akad. Nauk SSSR* Ser. Mat. 34 1970

[13] Deza, M.; Grishukhin, V. P.; Laurent, M. Extreme hypermetrics and L-polytopes. Sets, graphs and numbers (Budapest, 1991), 157–209, *Colloq. Math. Soc. Janos Bolyai*, 60, North-Holland, Amsterdam, 1992.

[14] Deza, M. M.; Laurent, M. *Geometry of cuts and metrics.* Algorithms and Combinatorics, 15. Springer-Verlag, Berlin, 1997.

[15] Dickson, T. J. A sufficient condition for an extreme covering of n-space by spheres. *J. Austral. Math. So*c. 8 (1968) 56–62.

[16] R. M. Erdahl, S. S. Ryshkov, The empty sphere. *Canad. J. Math.* 39 (1987), no. 4, 794–824.

[17] R.M. Erdahl, Oliver CLub Talk in Cornell (2000b), in preparation

[18] R.M. Erdahl K. Rybnikov, New Infinite Series of Perfect Forms and Big Delaunay simplexes in $\mathbb{Z}^n$, to appear the Ryshkov Volume of the Proceedings of the Steklov Mathematical Institute (2002)

[19] P. Erdös, P. M. Gruber, J. Hammer, *Lattice points.* Pitman Monographs and Surveys in Pure and Applied Mathematics, 39. Longman Scientific & Technical, Harlow; copublished in the United States with John Wiley & Sons, Inc., New York, 1989.

[20] Gruber, P. M.; Lekkerkerker, C. G. *Geometry of numbers.* Second edition. North-Holland Mathematical Library, 37. North-Holland Publishing Co., Amsterdam-New York, 1987

[21] C. Haase, G. Ziegler, On the maximal width of empty lattice simplices. Combinatorics of polytopes. *European J. Combin.* 21 (2000), no. 1, 111–119.

[22] A. Korkine, G. Zolotareff, Sur les formes quadratiques, *Math. Ann.* 6 (1873), 366-389..

[23] J. Martinet, *Les réseaux parfaits des espaces euclidiens.* (French) [Perfect lattices of Euclidean spaces] Mathématiques. Masson, Paris, 1996

[24] Ryshkov, S. S.; Baranovskii , E. P. C-types of n-dimensional lattices and 5-dimensional primitive parallelohedra (with application to the theory of coverings). Cover to cover translation of *Trudy Mat. Inst. Steklov* 137 (1976). *Proc. Steklov Inst. Math.* 1978, no. 4, 140 pp.

[25] S. S. Ryshkov, S. S. The perfect form A_n^k : existence of lattices with a nonfundamental partition simplex; existence of perfect forms that are not reducible in the sense of Minkowski to a form with identical diagonal coefficients. (Russian) Investigations in number theory, 2. *Zap. Naučn. Sem. Leningrad. Otdel. Mat. Inst. Steklov.* (LOMI) 33 (1973), 65–71. English translation in *J. Soviet Math.* 6 (1976), 672-676

[26] Ryshkov, S. S.; Baranovskii , E. P., Classical methods in the theory of lattice packings, *Russian Mathematical Surveys 3*4, 1979. p.1-68

[27] Ryshkov, S. S.; Baranovskii , E. P. Classification of arithmetical types of Delaunay simplexes in n-dimensional lattices for $n \leq 6$. *Proceedings of 1998 Voronoi Conference*, Kiev, (1998).

[28] Ryshkov, S. S; On the structure of a primitive parallelohedron and Voronoĭ's last problem. (Russian) Uspekhi Mat. Nauk 53 (1998), no. 2(320),161–162; translation in Russian Math. Surveys 53 (1998), no. 2, 403–405

[29] Voronoi G. F. Nouvelles applications des paramèters continus à la théorie des formes quadratiques, *J. Reine Angew. Math.*, **133**, (1908), 97-178, **134** (1908-09), 198-287, **136** (1909), 67-178.

[30] G. F. Voronoi, *Sobranie sočineniĭ v treh tomah*, [Collected works in three volumes] **vol. 2**, (in Russian), Kiev (1952). Introduction and notes by B.N.Delaunay.