

CORNELL UNIVERSITY MATHEMATICS DEPARTMENT SENIOR THESIS

# *Fractal Analysis: Extending the Domain*

A THESIS PRESENTED IN PARTIAL FULFILLMENT  
OF CRITERIA FOR HONORS IN MATHEMATICS

**Shawn Drenning**

May 2005

BACHELOR OF ARTS, CORNELL UNIVERSITY

THESIS ADVISOR(S)

Robert Strichartz  
Department of Mathematics

# 1 Spectral Decimation

## 1.1 Preliminaries

The goal of this paper is to extend some of the fractal analysis that has been developed on the Sierpinski Gasket ( $SG_2$ ) to new fractal domains. First we want to define a new fractal,  $SG_3$ . The first level of  $SG_3$  is shown below. The fractal is then generated in the same way as  $SG_2$ .

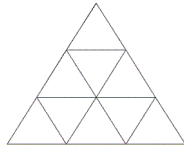


Figure 1: Building block for  $SG_3$

Just as with  $SG_2$  it is possible to define a Laplacian on each approximation to  $SG_3$  (and in the process define energy) and derive a pointwise formula for it. The pointwise formula for the Laplacian will be the same as the one for  $SG_2$  except for vertices where three cells meet. Here it is necessary to scale everything by  $2/3$ . This factor arises because when a function is integrated over  $SG_3$ , a vertex neighboring three cells is given  $3/2$  as much weight in the integral as a vertex neighboring only two cells. An easy computation shows that the energy renormalization factor is  $7/15$ . We also have a measure renormalization factor of  $1/6$ . Together, these will give a Laplacian renormalization factor of  $90/7$ . Finally, a Laplacian on  $SG_3$  can be defined as the renormalized limit of the Laplacian on level  $m$ . The first thing we want to study are eigenfunctions for this Laplacian. Our first step will be to derive a spectral decimation process similar to the one already defined on  $SG_2$ .

## 1.2 Eigenfunction extension algorithm

The first problem is derive an eigenfunction extension algorithm on  $SG_3$ . That is, given an eigenfunction on level  $m$ , we want to extend it to one on level  $m + 1$ . This is a local process and it is only necessary to solve the problem for one cell. If we can come up with an extension algorithm for Figure 2, we can extend it by linearity to get an algorithm for a cell with any values on the boundary.

If we want Figure 2 to be an eigenfunction for  $\lambda$ , we must satisfy the following four equations:

$$(4 - \lambda)x = 1 + x + y + w \tag{1}$$

$$(4 - \lambda)y = x + z + w \tag{2}$$

$$(4 - \lambda)z = y + z + w \tag{3}$$

$$(4 - \lambda)w = \frac{4}{3}(x + y + z) \tag{4}$$

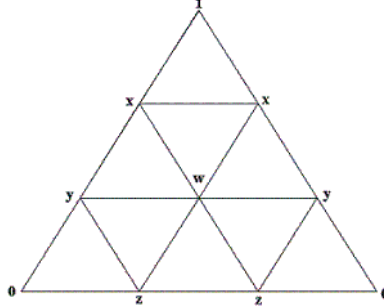


Figure 2: A function on one cell

If  $\lambda \neq 4$  then  $w = \frac{4}{3(4-\lambda)}(x+y+z)$ . Then add equations 1-3 and get

$$(4-\lambda)(x+y+z) = 1 + 2(x+y+z) + \frac{4}{4-\lambda}(x+y+z)$$

Using this we get:

$$x+y+z = \frac{4-\lambda}{4-6\lambda+\lambda^2}$$

Substitute this into equation 4 and when  $\lambda \neq 3 \pm \sqrt{5}$  obtain:

$$w = \frac{4}{3(4-6\lambda+\lambda^2)}$$

Now add  $y$  to both sides of equation 2 and use the equations derived for  $w$  and  $x+y+z$  and:

$$(5-\lambda)y = \frac{\frac{16}{3}-\lambda}{4-6\lambda+\lambda^2}$$

and thus when  $\lambda \neq 5$

$$y = \frac{16-3\lambda}{3(5-\lambda)(4-6\lambda+\lambda^2)}$$

Now subtract  $z$  from both sides of equation 3 and use the computed values for  $w$  and  $y$  and:

$$(3-\lambda)z = y+w = \frac{36-7\lambda}{3(5-\lambda)(4-6\lambda+\lambda^2)}$$

Thus, provided  $\lambda \neq 3$ , we have:

$$z = \frac{36-7\lambda}{3(3-\lambda)(5-\lambda)(4-6\lambda+\lambda^2)}$$

Similarly

$$x = \frac{1}{3-\lambda} + \frac{36-7\lambda}{3(3-\lambda)(5-\lambda)(4-6\lambda+\lambda^2)}$$

Also, when  $\lambda = 4$ ,  $x = y = w = \frac{-1}{3}$  and  $z = \frac{2}{3}$  satisfy the system of equations, so  $\lambda = 4$  is not a forbidden eigenvalue. From all of this, we get the following eigenfunction extension algorithm (in cases when  $\lambda \neq 5, 3$ , or  $3 \pm \sqrt{5}$ ):

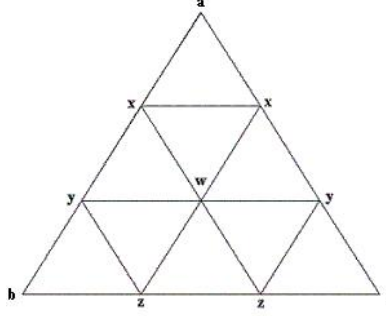


Figure 3: A general function on one cell

$$\begin{aligned} w &= \frac{4}{3(4-6\lambda+\lambda^2)}(a+b+c) \\ x &= \left(\frac{1}{3-\lambda} + \frac{36-7\lambda}{3(3-\lambda)(5-\lambda)(4-6\lambda+\lambda^2)}\right)a \\ &\quad + \frac{16-3\lambda}{3(5-\lambda)(4-6\lambda+\lambda^2)}b \\ &\quad + \frac{36-7\lambda}{3(3-\lambda)(5-\lambda)(4-6\lambda+\lambda^2)}c \end{aligned}$$

We get similar formulas for  $y$  and  $z$ .

### 1.3 Eigenvalues associated with the eigenfunction extension algorithm

The next step is to figure out which eigenvalues we can use with the eigenfunction extension algorithm. That is, if  $f_m$  is an eigenfunction on level  $m$  with eigenvalue  $\lambda_m$ , we want to find an eigenvalue  $\lambda_{m+1}$  such that when we extend  $f_m$  to  $f_{m+1}$  using  $\lambda_{m+1}$  in the eigenfunction extension algorithm,  $f_{m+1}$  is an eigenfunction for  $\lambda_{m+1}$ . The only vertices where  $f_{m+1}$  could possibly not satisfy the eigenfunction equations are vertices where two or three cells meet. We will solve the problem in the case of two cells meeting. It is easy to check that our solution also works for where three cells meet.

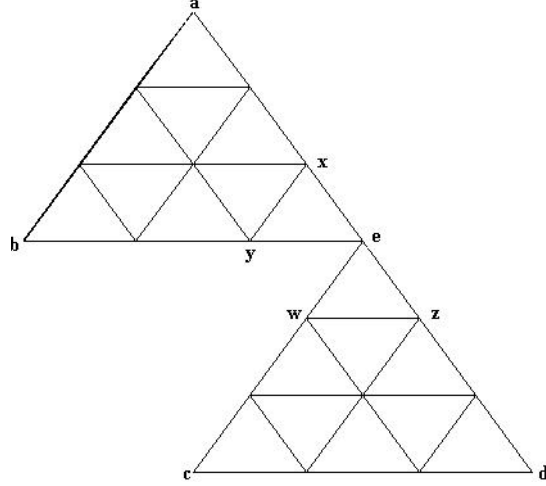


Figure 4: Two cells meeting

From Figure 4, we see that we need to find  $\lambda_{m+1}$  satisfying

$$(4 - \lambda_{m+1})e = w + x + y + z$$

Using the eigenfunction extension algorithm derived above, we can write  $x$ ,  $y$ ,  $z$ , and  $w$  in terms of  $\lambda_{m+1}$ ,  $a$ ,  $b$ ,  $c$ ,  $d$ , and  $e$ . Doing this, we get

$$\begin{aligned} x + y + z + w &= \frac{84 - 32\lambda_{m+1} + 3\lambda_{m+1}^2}{3(3 - \lambda_{m+1})(5 - \lambda_{m+1})(4 - 6\lambda_{m+1} + \lambda_{m+1}^2)}(a + b + c + d) \\ &\quad + 4\left(\frac{1}{3 - \lambda_{m+1}} + \frac{36 - \lambda_{m+1}}{3(3 - \lambda_{m+1})(5 - \lambda_{m+1})(4 - 6\lambda_{m+1} + \lambda_{m+1}^2)}\right)e \end{aligned}$$

Since  $a + b + c + d = (4 - \lambda_m)e$  we can rewrite this as:

$$\begin{aligned} (4 - \lambda_{m+1})e &= (4 - \lambda_m)\left(\frac{84 - 32\lambda_{m+1} + 3\lambda_{m+1}^2}{3(3 - \lambda_{m+1})(5 - \lambda_{m+1})(4 - 6\lambda_{m+1} + \lambda_{m+1}^2)}\right)e + \\ &\quad 4\left(\frac{1}{3 - \lambda_{m+1}} + \frac{36 - \lambda_{m+1}}{3(3 - \lambda_{m+1})(5 - \lambda_{m+1})(4 - 6\lambda_{m+1} + \lambda_{m+1}^2)}\right)e \end{aligned}$$

Then cancel the  $e$ 's<sup>1</sup> and simplify and we obtain:

$$\lambda_m = \frac{3(\lambda_{m+1} - 5)(\lambda_{m+1} - 4)(\lambda_{m+1} - 3)\lambda_{m+1}}{3\lambda_{m+1} - 14} \quad (5)$$

<sup>1</sup>We can assume we chose a point where  $e \neq 0$ ; otherwise we would be extending the constant function 0

In this last calculation, we cancel  $\lambda - 6$  from both sides so it is thus necessary that  $\lambda_{m+1} \neq 6$ . This gives us our last “forbidden” value. The following two graphs show the derived function and its inverse:

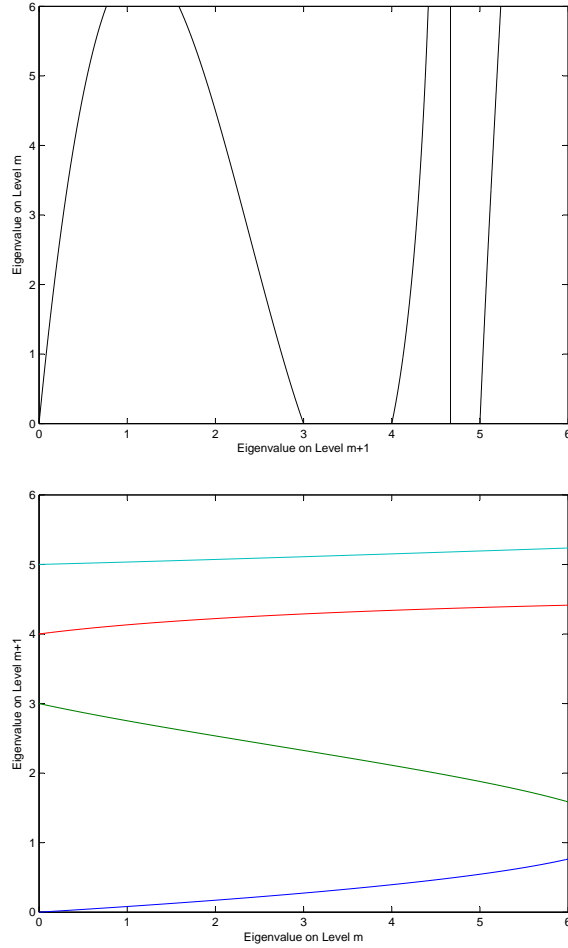


Figure 5: The graph of the function in equation 5 and its inverse

Recall that for the five forbidden eigenvalues  $3$ ,  $5$ ,  $3 \pm \sqrt{5}$ , and  $6$  that the eigenfunction extension algorithm will not work. If  $f$  is an eigenfunction on level  $m$  corresponding to eigenvalue  $\lambda$ , then using the inverse to formula 5 (this inverse can be explicitly computed), we get four potential eigenvalues to use with the eigenfunction extension formula in order to extend  $f$  to an eigenfunction on level  $m + 1$ . For each of these four potential eigenvalues that is not a forbidden value, we can use the eigenfunction extension algorithm to get an eigenfunction on level  $m + 1$ . From the second graph in Figure 5, it is easy to see that only the

eigenvalues 0 and 6 are mapped to forbidden values. The eigenvalue 0 maps to 0, 3, 4, and 5. Since 3 and 5 are forbidden eigenvalues, if  $\lambda = 0$ ,  $f$  can only be extended in two ways. Similarly, 6 maps to  $3 \pm \sqrt{2}$  and  $3 \pm \sqrt{5}$  and since  $3 \pm \sqrt{5}$  are forbidden values, if  $\lambda = 6$ ,  $f$  can only be extended in two ways. Finally, for any other value of  $\lambda$ ,  $f$  can be extended in four ways.

### 1.4 Eigenfunctions not arising from eigenfunction extension algorithm

For eigenvalues  $\lambda = 3, 5, 3 \pm \sqrt{5}$ , and 6, we saw that the eigenfunction extension algorithm did not work. However, there are eigenfunctions with these eigenvalues—they just cannot be obtained using the eigenfunction extension algorithm. We can still completely describe all eigenfunctions with these forbidden eigenvalues. We will do this for the particular cases of Dirichlet and Neumann eigenfunctions. First, we describe the Dirichlet eigenfunctions appearing on level 1. If the necessary equations are solved on level 1, the Dirichlet eigenfunctions illustrated in Figure 6 are found. These eigenfunctions can be rotated to give

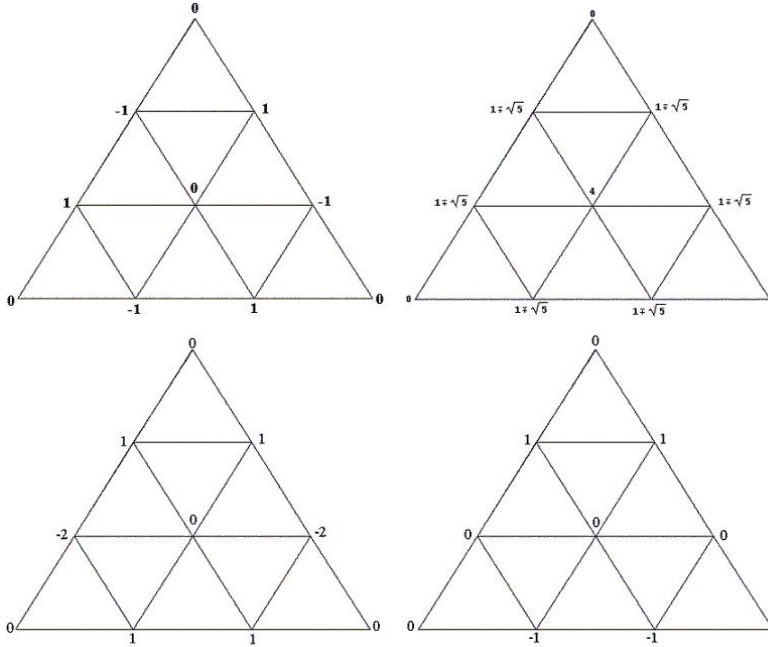


Figure 6: Dirichlet eigenfunctions on level 1 of  $SG_3$  for eigenvalues 6,  $3 \pm \sqrt{5}$ , 5, and 3

additional eigenfunctions. Doing this gives us 1, 1, 2 and 2 linear independent eigenfunctions for eigenvalues 6,  $3 \pm \sqrt{5}$ , 5, and 3 respectively.

There are no eigenfunctions of eigenvalue  $3 \pm \sqrt{5}$  on any level other than level 1. For eigenvalues 3, 5, and 6, there are Dirichlet eigenfunctions that are

“born” on each level. We can completely describe these eigenfunctions. In order to do this, first we need to consider the Neumann eigenfunctions on level 1. The best way to compute these is to separately find the symmetric and anti-symmetric eigenfunctions. First observe that the Dirichlet-6 eigenfunction on level 1 shown in Figure 6 is also a Neumann anti-symmetric eigenfunction. Additionally, we find the Neumann anti-symmetric eigenfunction shown in Figure 7. The Neumann symmetric eigenfunctions on level 1 are shown in Figure 8.

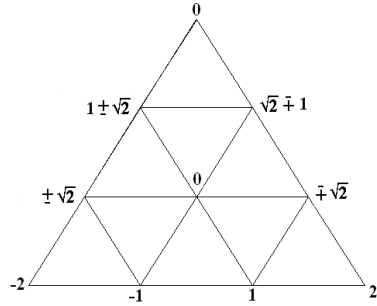


Figure 7: Neumann anti-symmetric eigenfunction for eigenvalues  $3 \pm \sqrt{2}$

After considering rotations, we see that together these Neumann eigenfunctions

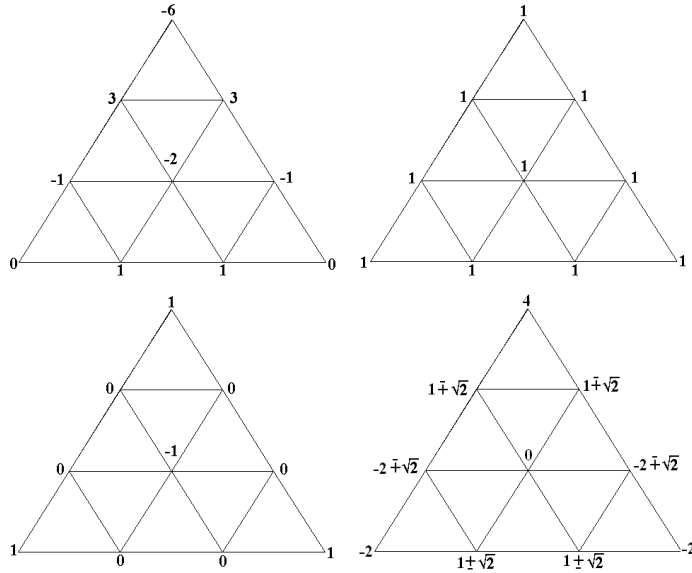


Figure 8: Neumann symmetric eigenfunctions for eigenvalues 6, 0, 4, and  $3 \pm \sqrt{2}$  generate a 10-dimensional space as desired. The multiplicities for eigenvalues 6,



0, 4, and  $3 \pm \sqrt{2}$  respectively are 4, 1, 1, and 4 (2 for each).

We can use the eigenfunctions we derived on level 1 to construct eigenfunctions with forbidden eigenvalues on higher levels. The constructions we will give are general; that is they work for any level of our approximation to  $SG_3$ . Furthermore, each eigenfunction will be supported in only a small part of the domain (this will be made more explicit later). For this reason, we can use the same procedure to construct Neumann and Dirichlet eigenfunctions. We just need to be careful near the boundary.

For eigenvalue 6, we can construct an eigenfunction that is equal to the Dirichlet-6 eigenfunction on one cell of level  $m$  and zero elsewhere. This will be both a Dirichlet and Neumann eigenfunction. It is clear that all of these eigenfunctions are linearly independent. Next, notice that the symmetric Neumann-6 eigenfunction can be rotated to give three linearly independent Neumann-6 eigenfunctions. Around any vertex  $v$  on level  $m$ , we can construct an eigenfunction on level  $m + 1$  that is equal to one of the symmetric Neumann-6 eigenfunctions on each cell containing  $v$  and equal to zero on each cell not containing  $v$ . Notice that each one of these eigenfunctions is non-zero on exactly one vertex of level  $m$ . Thus, it is easy to see that these eigenfunctions together with the first set are linearly independent. Also, each one of these eigenfunctions will be Neumann and all but the three that are non-zero on the boundary will be Dirichlet. Thus, on level  $m + 1$  we have  $T_m + V_m - 3$  Neumann and  $T_m + V_m$  Dirichlet eigenfunctions where  $T_m$  and  $V_m$  are respectively the number of cells and vertices on level  $m$ .

For eigenvalues 3 and 5, the first thing we need is an anti-symmetric Dirichlet eigenfunction on level 1. We already have symmetric Dirichlet eigenfunctions on level 1; if we reflect these eigenfunctions about an axis of non-symmetry and subtract the result, we get anti-symmetric eigenfunctions. The result is exhibited in Figure 9. Around any cycle of level  $m$  we can connect these eigen-

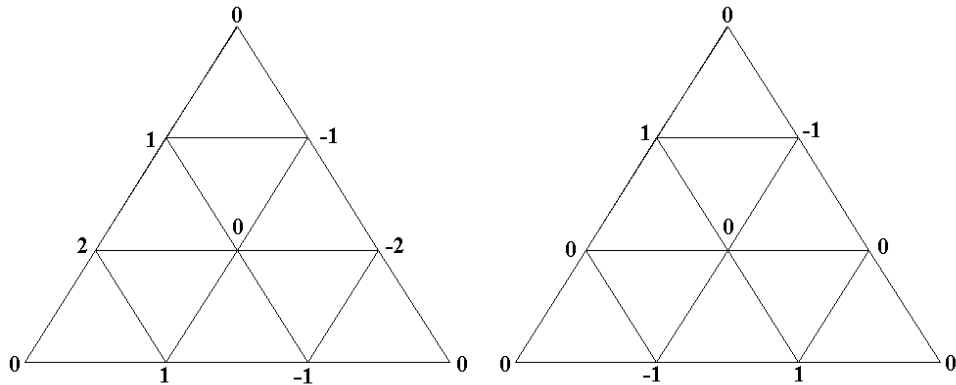


Figure 9: Anti-symmetric Dirichlet eigenfunctions for eigenvalues 3 and 5

functions like batteries to obtain a new eigenfunction. This concept is illustrated in Figure 10.

Thus, if  $C_m$  is the number of cycles on level  $m$ , on level  $m + 1$  we get

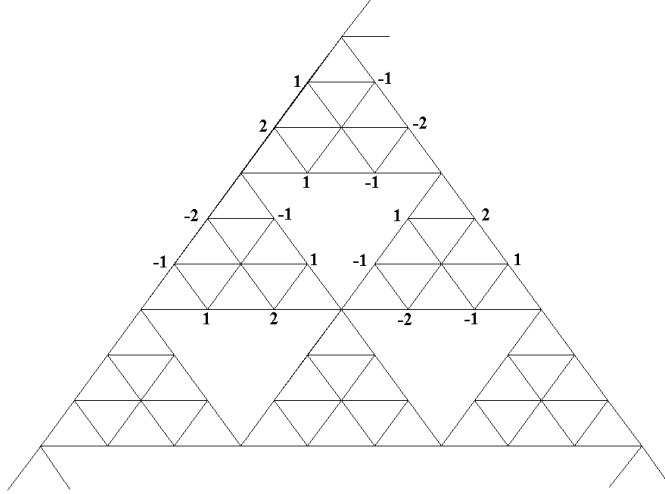


Figure 10: A eigenfunction for eigenvalue 3 obtained by connecting eigenfunctions like batteries

$C_m$  eigenfunctions each for eigenvalues 3 and 5. It can be checked that these eigenfunctions are linearly independent. Additionally, in the Dirichlet case, we can string the eigenfunctions in Figure 9 along the sides of the fractal to obtain two additional eigenfunctions. The multiplicities of all these new eigenfunctions are summarized in Table 1.

Table 1: Multiplicities of eigenvalues not coming from spectral decimation

$\lambda_{m+1}$	Dirichlet Multiplicity	Neumann Multiplicity
3 or 5	$C_m + 2$	$C_m$
6	$T_m + V_m - 3$	$T_m + V_m$

We know that on level  $m$  we should have  $V_m$  Neumann and  $V_m - 3$  Dirichlet eigenfunctions. Now we want to check that our spectral decimation process actually gives this many eigenfunctions. The first step is to give some simple recursive formulas for  $T_m$ ,  $C_m$ , and  $V_m$ . We easily see that:

$$T_m = 6T_{m-1} \tag{6}$$

$$C_m = C_{m-1} + 3T_{m-1} \tag{7}$$

$$V_m = V_{m-1} + 7T_{m-1} \tag{8}$$

Now we will verify that the spectral decimation gives all of the Dirichlet eigenfunctions. We will do this by induction on the level  $m$ . On level 1, we have

given all 7 (counting multiplicities) eigenfunctions. On level 2, we get 26 eigenfunctions by extending the 7 eigenfunctions on level 1 with the eigenfunction extension algorithm. Using Table 1, we see that we also get 5 eigenfunctions each for eigenvalues 3 and 5 and 13 eigenfunctions for eigenvalue 6. Together this gives us 49 Dirichlet eigenfunctions for level 2. It is easy to see that  $V_m - 3$  is indeed 49. Now assume that the spectral decimation process gives us  $V_m - 3$  Dirichlet eigenfunctions on level  $m$  (with  $m > 3$ ). We want to show that we get  $V_{m+1} - 3$  eigenfunctions on level  $m + 1$ . From Table 1, we know that on level  $m + 1$  we will have, in total,  $2C_m + T_m + V_m + 1$  eigenfunctions of eigenvalues 3, 5, or 6. Next note that there will never be any Dirichlet eigenfunctions with eigenvalue 0. Thus, for every eigenfunction on level  $m$  not of eigenvalue 6, we will get four eigenfunctions on level  $m + 1$  from the eigenfunction extension algorithm. For each eigenfunction of eigenvalue 6, we will get two eigenfunctions on level  $m + 1$ . Since there are  $T_{m-1} + V_{m-1} - 3$  eigenfunctions of eigenvalue 6 on level  $m$ , we get  $4V_m - 12 - 2(T_{m-1} + V_{m-1} - 3) = 4V_m - 2T_{m-1} - 2V_{m-1} - 6$  eigenfunctions on level  $m + 1$  from the eigenfunction extension algorithm. Thus, we want to show that:

$$V_{m+1} - 3 = (2C_m + T_m + V_m + 1) + (4V_m - 2T_{m-1} - 2V_{m-1} - 6)$$

Replacing  $V_{m+1}$  with  $V_m + 7T_m$  and collecting terms we get

$$6T_m = 4V_m + 2C_m - 2V_{m-1} - 2T_{m-1} - 2$$

Now use the recursive formulas for  $T_m$ ,  $V_m$ , and  $C_m$  to rewrite this as

$$36T_{m-1} = 4V_{m-1} + 28T_{m-1} + 2C_{m-1} + 6T_{m-1} - 2V_{m-1} - 2T_{m-1} - 2$$

Simplified this becomes

$$2T_{m-1} = V_{m-1} + C_{m-1} - 1$$

This final formula is easy to prove for all  $m$  by induction. This completes the proof and shows that the spectral decimation method does indeed give all of the Dirichlet eigenfunctions. A similar computation shows that spectral decimation will give all of the Neumann eigenfunctions.

## 1.5 Spectral Decimation on New Gaskets

We now have a working spectral decimation method for two self-similar fractals:  $SG_2$  and  $SG_3$ . Our next step is to show how we can use this to derive a spectral decimation method on a certain class of non-self-similar gaskets. Given a sequence  $(b_1, b_2, \dots)$  of 2's and 3's, we can define a new fractal in an obvious way; at each stage, if  $b_i = 2$  replace each triangle with the level 1 approximation to  $SG_2$  and if  $b_i = 3$  use the  $SG_3$  approximation. Thus,  $(2, 2, \dots)$  corresponds to  $SG_2$  and  $(3, 3, \dots)$  corresponds to  $SG_3$  and any other sequence corresponds to something inbetween.

Just as with  $SG_2$  and  $SG_3$ , it is possible to define a Laplacian on a approximation to any of these new gaskets. It can be verified that this new Laplacian will have the same pointwise formula as either  $SG_2$  or  $SG_3$  depending on whether  $b_m$  is 2 or 3.<sup>2</sup> For instance, the gasket corresponding to  $(2, 3, 2, 3, \dots)$  will have the same pointwise Laplacian formula as  $SG_2$  on levels  $1, 3, 5, \dots$  and  $SG_3$  on levels  $2, 4, 6, \dots$ . Finally, by taking the limit of these pointwise formulas, it is possible to define a Laplacian on the actual gasket. At each stage the Laplacian renormalization factor for  $SG_2$  or  $SG_3$  is used depending on what  $b_i$  is.

Now that we have a well-defined Laplacian on this new class of gaskets, a natural thing to do is to study eigenfunctions of this Laplacian. In fact, using our knowledge of spectral decimation on  $SG_2$  and  $SG_3$ , it is possible to develop a spectral decimation method on any of these gaskets. The main idea here is that all the processes involved in the spectral decimation for  $SG_2$  and  $SG_3$  were local. Our first step is to define an eigenfunction extension formula. Recall that in deriving the eigenfunction extension formula for  $SG_2$  and  $SG_3$  it was only necessary to work inside one cell of the gasket. Similarly, in computing the eigenvalues associated with eigenfunction extension formula, it was only necessary to examine two (or for certain vertices of  $SG_3$  three) neighboring cells. So extending eigenfunctions is an entirely local process. Thus, for a more general gasket we can utilize the eigenfunction extension formulas we already know for  $SG_2$  and  $SG_3$ . In particular, if  $b_{m+1} = 2$  (respectively  $b_{m+1} = 3$ ) we can use the eigenfunction extension formula for  $SG_2$  (respectively  $SG_3$ ) to extend an eigenfunction on level  $m$  to one on level  $m + 1$ . When extending an eigenfunction from level  $m$  to level  $m + 1$ , the forbidden eigenvalues will correspond to the forbidden eigenvalues for  $SG_2$  or  $SG_3$  depending on whether  $b_{m+1}$  is 2 or 3.

The next step is to examine the eigenfunctions on these general gaskets that do not arise from the eigenfunction extension algorithm. Recall that for both  $SG_2$  and  $SG_3$ , we were able to come up with explicit formulas in terms of  $C_m$ ,  $T_m$ , and  $V_m$  for the number of eigenfunctions that are “born” on level  $m + 1$ . To derive these formulas, recall that only the structure of the fractal on levels  $m$  and  $m + 1$  was used. Furthermore, the only things used about the structure of the fractal on level  $m$  were the values for  $C_m$ ,  $T_m$ , and  $V_m$ . For this reason, in the case of a more general gasket, the same formulas can be used as were used for  $SG_2$  and  $SG_3$ . If  $b_{m+1} = 2$ , use the  $SG_2$  formulas and if  $b_{m+1} = 3$ , use the  $SG_3$  formulas. By counting them, it can be verified that, together with the eigenfunction extension formulas, this gives the correct number of eigenfunctions for level  $m + 1$ . Thus, we have a working spectral decimation method for this more general class of gaskets.

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<sup>2</sup>Intuitively this makes sense since at any vertex on level  $m$ , locally it is not possible to determine  $b_i$  for  $i < m$ .

## 1.6 Spectral Gaps

Now we want to study the spectrum of the Laplacian on these general gaskets. The two things we will demonstrate are that, in general, there are always gaps in the spectrum of the Laplacian and that, in certain cases, the set of ratios of eigenvalues has gaps.

It is easy to see by examining the spectral decimation method that all eigenfunctions on any of these gaskets will have eigenvalues between 0 and 6 (ignoring renormalization for now). All of the “born” eigenfunctions satisfy this criteria and it is easy to verify that for both  $SG_2$  and  $SG_3$  that the eigenfunction extension formula extends eigenfunctions with eigenvalues in  $[0, 6]$  to ones with eigenvalues in  $[0, 6]$  (for  $SG_3$  see Figure 5). Now we want to show that there are gaps in the spectrum of eigenvalues on level  $m$ . All of the eigenvalues on level  $m$  will lie in the image of the inverse of the appropriate decimation function applied to the interval  $[0, 6]$   $m - 1$  times plus a finite number of exceptions arising from the eigenfunctions born on levels  $2, 3, \dots, m$ . Now notice that the inverse to the decimation function for  $SG_2$  and for  $SG_3$  has gaps in its range. More precisely, the inverse decimation functions for  $SG_2$  together map  $[0, 6]$  to  $[0, 2] \cup [3, 5]$  and the inverse decimation functions for  $SG_3$  together map  $[0, 6]$  to  $[0, 3 - \sqrt{5}] \cup [3 - \sqrt{2}, 3] \cup [4, 3 + \sqrt{2}] \cup [5, 3 + \sqrt{5}]$ . Thus, with each iteration of these functions, more gaps are introduced and it is clear that there will be gaps in the spectrum on level  $m$ . From this it follows that in the limit the spectrum will have gaps too. In fact, it is easy to see that the spectrum will resemble a Cantor set.

Now we will outline an algorithm to find gaps in the ratios of eigenvalues. We will outline this algorithm for the special case of Dirichlet eigenfunction on  $SG_3$ , but it will be clear how it can be generalized to the Neumann case and for any of the more general gaskets we have studied.

First, label the four branches of the inverse decimation function from bottom to top (see Figure 5)  $R_{31}^{-1}$ ,  $R_{32}^{-1}$ ,  $R_{33}^{-1}$ , and  $R_{34}^{-1}$ . Now we need to think about how eigenvalues for the Laplacian are obtained. First, we start on some level with an eigenvalue  $\lambda$  associated to an eigenfunction born on that level. Then, in some order, the four functions  $R_{31}^{-1}$ ,  $R_{32}^{-1}$ ,  $R_{33}^{-1}$ , and  $R_{34}^{-1}$  are repeatedly applied to  $\lambda$ . Recall that at each stage we also have to multiply by the renormalization factor of  $90/7$ , so in order to get a finite limit, it is necessary that the unrenormalized limit be 0. Thus, we can only apply  $R_{32}^{-1}$ ,  $R_{33}^{-1}$ , and  $R_{34}^{-1}$  finitely many times and there must exist a point after which we only apply  $R_{31}^{-1}$ .

Now we can outline a procedure for finding gaps in the ratios. The first step is to find a cover for the spectrum of eigenvalues (ignoring renormalization factors). We can start with  $[0, 6]$  and refine it in the same we did to find gaps in the spectrum: repeatedly apply the four inverse decimation functions. We then need to add a finite number of values corresponding to the “born” eigenfunctions. The more we refine our cover, the more gaps in the ratios we can hope to find. Every eigenvalue (except the one obtained by starting with  $3 - \sqrt{5}$  on level 1 and repeatedly applying  $R_{31}^{-1}$ ) can be obtained by repeatedly applying  $R_{31}^{-1}$  to an eigenvalue  $\lambda$  of the level  $m$  approximation with the condition

that  $\lambda$  is not in the range of  $R_{31}^{-1}$ . Thus, we can ignore the part of the spectrum in the range of  $R_{31}^{-1}$ . Call our cover with this portion removed  $C$ . Now we compute all possible ratios that can arise by starting with two eigenvalues  $\lambda_1$  and  $\lambda_2$  in  $C$ , going to the limit by repeatedly applying  $R_{31}^{-1}$  to  $\lambda_1$  and  $\lambda_2$  (and renormalizing), and then taking the ratio. Let the  $m_1$  and  $m_2$  respectively be the last level  $\lambda_1$  and  $\lambda_2$  are not in the range of  $R_{31}^{-1}$ . Observe that if we replaced  $m_1$  with  $m_1 + 1$ , the ratio will just change by a factor of  $90/7$ . Thus, except for a few exceptions (we have to be careful with the eigenvalues that come from the eigenfunctions on level 1 with eigenvalue  $3 \pm \sqrt{5}$  since  $3 \pm \sqrt{5}$  never again appears as an eigenvalue), if  $r$  is a ratio of eigenvalues, so is  $90/7r$  and  $7/90r$ . Also, it is easy to see that if  $r$  is a ratio, then  $1/r$  will be one too. For this reason, it is enough to look for gaps in the interval  $[1, 90/7]$ . Also, since we are dividing the eigenvalues, the actual values of  $m_1$  and  $m_2$  are unimportant; all we need to know is  $m_1 - m_2$ . Then, it is easy to see that since we only care about ratios in  $[1, 90/7]$  that we can assume that  $|m_1 - m_2| \leq 1$ . Finally, we do not actually know all the eigenvalues we want to take ratios of; we just know intervals they lie in. However, given two intervals, it is an easy exercise to compute all ratios of eigenvalues in those intervals (just work with the endpoints).

Now we have all the necessary tools for an algorithm for computing gaps in the spectrum of ratios. First we take a suitable cover made up of intervals and points that all our eigenvalues (unrenormalized) lie in. Next, for any two intervals (and/or points) in this collection, we compute all possible ratios (in  $[0, 90/7]$ ) of eigenvalues obtained by applying  $R_{31}^{-1}$  to possible eigenvalues (for the approximation) in these intervals (and/or points). This provides us with a cover of the spectrum of ratios and we can then easily check for gaps. While some of the details of this algorithm are missing, it should be clear that this is a feasible way to compute gaps in the ratios and this method could easily be implemented on a computer (and, in fact, it has been). For more details, see [www.math.cornell.edu/~sld32](http://www.math.cornell.edu/~sld32).

One important thing about this algorithm is that it cannot be used to conclusively say gaps do not exist. That is, any gaps it finds do exist, but just because it does not find any gaps, does not mean they are not there. We attempted to use this algorithm to show that any of the general gaskets we have studied have gaps in the spectrum of the ratios. However, the computations overwhelmed the computer before gaps could be found in all cases<sup>3</sup>. In many cases, though, we were able to find gaps. It can be worked out by hand that  $SG_2$  has gaps in the spectrum of ratios. Using a computer it can be verified that  $SG_3$  also has gaps in the spectrum. Also, any gasket corresponding to a sequence of  $2$ 's and  $3$ 's that can be separated into blocks of 23 and 32 will have gaps in the spectrum of ratios. Finally, in many other specific cases, gaps were found in the spectrum of ratios.

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<sup>3</sup>In large part because as the cover of the spectrum of eigenvalues is refined, the number of intervals that have to be considered grows exponentially

## 2 Miscellaneous Results and Calculations

### 2.1 An Integral Calculation

The goal of this section is to derive a formula for the norm of an eigenfunction on  $SG_3$ . In general, we know the norm of function on level  $m + 1$  is given by

$$\|f\|_{m+1}^2 = \sum_{x \in V_{m+1}} w(x)|f(x)|^2$$

where  $w(x)$  is the number of cells  $x$  is in. We can rewrite this formula as:

$$\|f\|_{m+1}^2 = \|f\|_m^2 + \sum_{x \in V_{m+1} \setminus V_m} w(x)|f|^2$$

Now, using the eigenfunction extension formula, we want to rewrite the second summand. Recall the weights for the eigenfunction extension formula on  $SG_3$

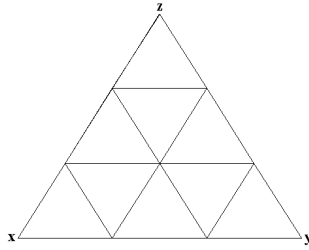
$$a = \frac{1}{3 - \lambda} + \frac{36 - 7\lambda}{3(3 - \lambda)(5 - \lambda)(4 - 6\lambda + \lambda^2)} \quad (9)$$

$$b = \frac{16 - 3\lambda}{3(5 - \lambda)(4 - 6\lambda + \lambda^2)} \quad (10)$$

$$c = \frac{36 - 7\lambda}{3(3 - \lambda)(5 - \lambda)(4 - 6\lambda + \lambda^2)} \quad (11)$$

$$d = \frac{4}{3(4 - 6\lambda + \lambda^2)} \quad (12)$$

Now we will work within one cell. Let that cell have values  $x, y$ , and  $z$  on the boundary.



Now, using the eigenfunction extension formula, we can compute the values on the interior of the cell in terms of the values on the boundary. The value of the middle vertex will be

$$(f(x) + f(y) + f(z))d$$

If we square this we get

$$d^2(f(x) + f(y) + f(z)) + 2d^2(f(x)f(y) + f(y)f(z) + f(x)f(z))$$

Next we compute the value on one of the other vertices. This will be like

$$af(x) + bf(y) + cf(z)$$

The sum of the squares of values on the interior vertices is easily computed as:

$$(2a^2 + 2b^2 + 2c^2)(f(x)^2 + f(y)^2 + f(z)^2) + (4ab + 4ac + 4bc)(f(x)f(y) + f(y)f(z) + f(x)f(z))$$

Note that  $w$  is 3 for the middle vertex and 2 for the other ones. This gives us

$$(4a^2 + 4b^2 + 4c^2 + 3d^2)(f(x)^2 + f(y)^2 + f(z)^2) + (8ab + 8ac + 8bc + 6d^2)(f(x)f(y) + f(y)f(z) + f(x)f(z))$$

for the weighted sum of  $|f|^2$  restricted to this one cell. Now sum this over all cells and

$$\begin{aligned} \sum_{x \in V_{m+1} \setminus V_m} w(x)|f(x)|^2 &= (4a^2 + 4b^2 + 4c^2 + 3d^2)\|f\|_m^2 & (13) \\ &+ (8ab + 8ac + 8bc + 6d^2) \sum_{x \sim y} f(x)f(y) & (14) \end{aligned}$$

Next we need to compute  $\sum_{x \sim y} f(x)f(y)$ . To do this we will use the weak formulation for the Laplacian.

$$\sum_{x \sim y} (f(x) - f(y))^2 = E_m(f, f) = - \int \Delta f f d\mu$$

But  $f$  is an eigenfunction, so  $\Delta f = -\lambda_m f$ . Make this substitution and

$$\sum_{x \sim y} (f(x) - f(y))^2 = \lambda_m \int f^2 d\mu$$

Expand  $(f(x) - f(y))^2$  and replace  $\int f^2 d\mu$  with  $\|f\|_m^2$ . Simplified, this becomes

$$-2 \sum_{x \sim y} f(x)f(y) + 2\|f\|_m^2 = \lambda_m \|f\|_m^2$$

and finally:

$$\sum_{x \sim y} f(x)f(y) = \frac{(2 - \lambda_m)\|f\|_m^2}{2}$$

Now substitute this into equation 14 and also use the eigenvalue extension algorithm to write  $\lambda_m$  in terms of  $\lambda_{m+1}$ . Simplified this gives the following formula for the norm of  $f$  on level  $m+1$  in terms of the norm on level  $m$ .

$$\|f\|_{m+1}^2 = 2\|f\|_m^2 \frac{-11760 + 60200\lambda_{m+1} - 79654\lambda_{m+1}^2 + 47267\lambda_{m+1}^3 - 14905\lambda_{m+1}^4 + 2605\lambda_{m+1}^5 - 239\lambda_{m+1}^6 + 9\lambda_{m+1}^7}{3(\lambda_{m+1} - 5)^2(\lambda_{m+1} - 3)^2(3\lambda_{m+1} - 14)(4 - 6\lambda_{m+1} + \lambda_{m+1}^2)} \quad (15)$$



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